Unpacking Division
to Build Teachers’ Mathematical Knowledge

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Accepted for Publication in Teaching Children Mathematics
November 2004

Note: This article is based upon work supported by the National Science Foundation under Grant No. EHR-0314898. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.
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“First I broke apart 169 into 140 and 29 and than I divided 140 by 14,” explained Andreau. Whereas Kendra commented that she also used 140, but began by multiplying 10 x 14. These teachers are describing the approaches they used to solve 169 ÷ 14. They were challenged to generate two approaches for solving this multidigit division problem without using the conventional division algorithm. Upon examining their work, we were struck by the various ways in which they approached the task and by the operations encompassed within their solution strategies. We were intrigued when they became stuck in their thinking and when they gained insights into the current boundaries of their division knowledge.

Based on our experiences in working with both prospective and practicing teachers, we have found the need to support teachers in developing a much deeper understanding of number and operations than most now hold, particularly for division. Our knowledge as adults is often compressed (Ball and Bass 2000). In fact, we have worked for years to compact and streamline our mathematical knowledge. This compression of knowledge is central to the discipline of mathematics. However, “knowing mathematics sufficiently for teaching requires being able to unpack ideas and make them accessible as they are first encountered by the learner, not only in their finished form” (Ball 2003, p. 4). Prospective teachers enter our courses with compressed knowledge of division. They are able to demonstrate procedural or algorithmic skill with division of multidigit numbers, but are limited in their conceptual understanding which was long ago compacted or perhaps never even known.

The intent of this article is to examine the unpacking of the mathematical knowledge needed for teaching division. What does one need to know and understand to teach division well? What comprises the package (Ma 1999) or bundle (Ball and Bass 2000) of knowledge—the key ideas, understandings, connections, and sensibilities—that teachers need to develop so that it is available for teaching? How can we, as teacher educators, reveal the complexities of division in ways that support teacher learning? We first present a core task for surfacing and unpacking one’s division knowledge and than discuss the understandings that might comprise a package of teacher knowledge for division.

Core Task

The purpose of a core task is to surface teachers’ current knowledge and understandings and to provide a context for grounding discussions over several class sessions. One task we have used productively to probe what teachers know about division is to ask them to generate strategies for solving multidigit division problems. The specific task they were given was:

The purpose of this task is to begin to think more deeply about division, so put your thoughts, attempts, and missteps all on paper. Solve the following division problems using two strategies other than the conventional division algorithm. Then solve the problems using the algorithm. Explain and represent your thinking using symbols, words, and diagrams, as appropriate.

169 ÷ 14

3480 ÷ 36
As recommended by CBMS (2001) in their report *The Mathematical Education of Teachers*, the key to turning prospective teachers “into mathematical thinkers is to work from what they do know—the mathematical ideas they hold, the skills the possess, and the contexts in which these are understood—so they can move from where they are to where they need to go” (p. 17). We have found that reflection upon and discussion of one’s own mathematical work to be a valuable teaching tool in building one’s mathematical knowledge for teaching.

The division problems in this core task were purposely selected to establish a basis on which to connect and deepen one’s knowledge. We chose $169 \div 14$ because of its potential to prompt use of multiples of 10 and because it can be solved readily using direct modeling or repeated subtraction. We chose $3480 \div 36$ to prompt more complex strategies as the numbers do not lend themselves well to using a multiple of 10. We also chose problems with remainders to see how this may impact solution strategies.

By posing this task, we were hoping to accomplish several objectives. First, we wanted to have insight into teachers’ grasp of number sense and operation sense. In particular, we wanted to see if they could demonstrate relationships among the operations. Second, we knew they could carry out the conventional division algorithm, but we wondered how deeply they understood it and whether they realized its complexity. We planned to use their solution strategies to tie together and connect pieces of mathematical knowledge while generating an appreciation of the complexity of the algorithm. Finally, we wanted the teachers to increase their own flexibility in solving multidigit division problems by developing the ability to use a variety of computational strategies.

**Reviewing and Selecting Work Samples**

Our general practice is to assign the task as homework two class sessions prior to discussing it. This gives us time to carefully study and reflect upon the work. We review the papers once looking for certain themes that might surface. Questions we consider during the first review include: How successful was the class in devising alternative strategies that demonstrate their understanding of division? Is there a “preferred” alternate strategy demonstrated by the class as a whole? What collective mathematical struggles or insights surfaced? Were there surprises or curiosities?

A second look through the papers encourages a more careful review. How fluent are their alternative strategies? How accurate is the mathematical notation? Are individuals able to express their thinking in writing as well as symbols? Are there solid connections between procedural and conceptual knowledge? Did specific individuals demonstrate mathematical insights or misconceptions in their work? What “windows” are offered into an individual’s thinking or struggles?

Finally, we select specific work samples for class discussion and examination. In the past, we have written the strategies on the chalkboard or made overhead transparencies. Our preference now is to put the selected strategies on individual chart paper. This allows us the opportunity to discuss each strategy on its own as well as being able to place strategies side by side for comparing and contrasting. We leave it up to the individuals to decide if they would like their name attached to their strategy or if they wish to remain anonymous.

**Unpacking Division through Examining Strategies**

Our goals for discussion included a careful examination of the embedded mathematics in the various strategies and the use of mathematical notation. We also wanted to give the teachers
opportunities to practice providing explanations, following the logic of each other’s strategies, and using the various strategies on new problems. The task, for many, provided an eye-opening experience as they worked with numbers in ways they had never before considered. The reactions of the prospective teachers ranged from “I had never thought division problems could be solved in so many different ways” to “I couldn’t get any ways to work successfully.”

To bridge this range of knowledge, we provided a structured process for examining the selected strategies. First we displayed a selected work sample. Then in pairs, teachers took turns explaining what they saw occurring in the algorithm and paraphrasing the others’ explanation. This created the foundation for an in-depth whole-class discussion of the embedded mathematics and how the alternate strategy was connected to the original problem.

We now briefly present some of the strategies for $169 \div 14$ to provide a glimpse into our discussions of the work samples and the surfacing of mathematical ideas. Maria’s work is shown in fig. 1. She represented 169 by drawing a picture of base ten blocks showing 16 tens and 9 units. First she distributed or dealt out the tens equally among 14 groups. Maria stated that she knew 14 tens was equivalent to 140 and that she then had 2 tens and 9 ones left to distribute. Next she broke apart the remaining 2 tens into 20 ones and distributed the ones equally among the 14 groups. An idea that emerged from discussion of this work was the meaning of division in partitive situations.

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169 ÷ 14 = □
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![Figure 1. Dealing out 169 into 14 groups.](image)

Next we compared this to the work in fig. 2 that shows the use of repeated subtraction and an interpretation of division as a measurement situation. Many of the teachers were surprised by this approach as they had never established a relationship between division and subtraction prior to this task. A particular highlight of this discussion was the identification and comparison of the meaning of the numbers 169, 14, 12, and 1 in each situation and why the meanings differed. In this case we found it helpful to create a real world context for both the partitive and measurement models to push for deeper understanding. It was intriguing to many that interpretations could be so different and yet still be represented by the same equation.
Kendra’s work (see fig. 3) illustrates the use of a missing factor approach. She explained, “I started by thinking about a multiplication problem using 14 that would get me close to 169. I used 10 because it’s easy to multiply, 10 × 14 = 140. That gives me 10 groups of 14.” Then she explained how she added on additional groups of 14 until she got as close to 169 as she could without going over. This discussion encouraged a closer look at the connections between multiplication and division and pushed us to examine the belief that “division is the opposite of multiplication.” This provided the basis for highlighting division as the inverse of multiplication as well as for making a connection to ways addition could be used as a related operation. It was noted that Kendra reasoned with a measurement interpretation of division by finding the number of groups of size 14. This lead to an interesting comparison to others in the class who used a missing factor approached but reasoned with a partitive interpretation of division.

In fig. 4 Andreau partitioned the dividend into 140 + 29. Her reasoning behind this first step was to “break apart” the dividend into two numbers with one being a number easily divisible by the divisor. Then she combined the partial quotients. This work prompted a discussion of
decomposition of numbers and properties of the operations. The question emerged, “If one can partition the dividend and it works, why can’t one partition the divisor?” Several individuals had attempted to do just this and found that it did not seem to work. These individuals had partitioned 14 into 10 + 4 and then divided 169 by 10 and then divided 169 by 4. The ensuing discussion engaged small groups in development of arguments as to whether or not one could partition the divisor. While some groups tried to generate a formal proof using properties of the operations, the situating of the problem in a real world context proved more convincing. For example, assume you had 169 stickers to share among 14 students, 10 girls and 4 boys. If you decompose the divisor you would be sharing 169 stickers among 10 girls and then sharing 169 stickers among 4 boys, thus creating a different situation.

Figure 4. Partitioning the dividend into 140 + 29

A Package of Division Knowledge for Teachers

In order to identify the key pieces that should comprise a package of division knowledge for teachers, we reviewed reports including The Mathematical Education of Teachers (CBMS 2001), Adding It Up (NRC 2001), and Principles and Standards for School Mathematics (NCTM 2000), examined textbooks in the field, and reflected upon our experiences with teachers and elementary school students. While we believe it is not possible to create a definitive list of topic, this gave us a starting point. In identifying topics, we attempted to discern the unique subject matter knowledge needed by teachers by examining how the mathematical knowledge would be used in the practice of teaching (Ball and Bass 2000).

The topics in fig. 5 adjacent to the word division (shaded in blue) formed our initial list. It is important for teachers to understand division as the inverse of multiplication as well as relationships to addition and subtraction. Teachers also need to be familiar with interpretations, models, representations, and applications of division. Additionally properties of the operations and decomposition of numbers appear to be important knowledge pieces.

From this, we added the topics around the outside (shaded in yellow) as a result of our interactions with prospective teachers and the knowledge that emerged as important for this particular group, such as giving meaning to the numbers and mathematical notation. What was striking to us as this package emerged was that our initial topics were phrased as declarative statements. This is the knowledge we wanted the prospective teachers to learn. While the topics that we added reflected action statements more reflective of mathematical knowledge needed for use in teaching.
Another important understanding for teachers is the connectedness of knowledge. Teachers “need to see the topics they teach as embedded in rich networks of interrelated concepts, know where, within those networks, to situate the tasks they set their students and the ideas these tasks elicit” (CBMS 2001, p. 55). It is the connections among these ideas, concepts, and procedures that enable teachers to portray mathematics as a unified body of knowledge rather than as isolated topics (Ma 1999). Knowing and being sensitive to all of these components of the package is the work of becoming a teacher of mathematics. One will than be more able to draw upon and make strategic use of these ideas in the practice of teaching (Ball 2003; Ball and Bass 2000). We view this package of division knowledge as a work in progress and plan to continue refining the package through further inquiry and work with teachers.

<table>
<thead>
<tr>
<th>Use and understand varied strategies and algorithms</th>
<th>Flexible Decomposition of Numbers</th>
<th>Division as the inverse of multiplication</th>
<th>Relationships among addition, subtraction, multiplication, and division</th>
<th>Connect informal and formal mathematical language and notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Establish equivalence of representations and strategies</td>
<td>Models for division (e.g., set, area, array, linear, branching)</td>
<td>DIVISION</td>
<td>Interpretations of division (e.g., measurement, partitive, combinations)</td>
<td>Give meaning to the numbers (e.g., different or same units)</td>
</tr>
<tr>
<td>Judge accuracy of strategies and potential for generalization</td>
<td>Representations (e.g., physical, geometric, symbolic)</td>
<td>Properties of the Operations</td>
<td>Applications and real world situations for division</td>
<td>Understand varied meanings of remainders</td>
</tr>
<tr>
<td>Use multiplicative relationships</td>
<td>Visualize actions and situations</td>
<td>Pose situations with varied problem structures</td>
<td>Identify use of place value</td>
<td>Grasp of division by zero as undefined</td>
</tr>
</tbody>
</table>

Figure 5. Example of a Package of Division Knowledge for Teachers

Key Knowledge Pieces

Within any package of knowledge, there are “key” pieces that hold more significance than others (Ma, 1999). The most important understandings for these prospective teachers were the relationships among the operations. As one teacher noted, “I am amazed that multiplication, subtraction, and addition can all help me get to the answer.” Another individual, more representative of the group commented, “I think I always understood the connections between all the operations and division, but I have never actually used them to solve division problems before.” Ma (1999) described the relationships among the four basic operations as the “road system” connecting all of elementary mathematics. We were surprised by the initial inclination of some teachers to not use these relationships in their strategies. Eventually the use of related operations, particularly multiplication, became the strategy of choice among this group of teachers.

Another important understanding was a deeper connection to the meaning of division. Most of the prospective teachers admitted they could do the conventional division algorithm, but they remarked that they “never really understood what was going on or even how to explain it” and that “coming up with other ways to solve division problems helped to see what division really means.”

A related key sensibility was an inclination to visualize actions and situations for division as reflected in the comment, “I started to look at division like I look at multiplication—seeing things in groups.” This supported the ability to identify the meaning of the numbers and to follow the logic of each other’s strategies.
Finally, the prospective teachers developed a greater appreciation of varied strategies and of the connection between their own mathematical knowledge and their future role as classroom teachers. As one reflected, “Seeing others’ points of view and different ways of thinking will help me in realizing that although I understand something one way, students may not.”

**Closing Comments**

The mathematical knowledge needed for teaching differs from what is needed for other occupations in that the knowledge must be usable in the practice of teaching whether selecting instructional tasks, representing ideas, orchestrating a class discussion, or evaluating students’ verbal or written responses (Ball 2003). This requires the unpacking of one’s compressed knowledge and bringing it to the surface for examination.

One approach to surfacing this knowledge is by engaging teachers in core tasks and then using their own work as sites for discussion. In particular, we have found that asking teachers to generate their own alternative computational strategies productive in revealing strengths and gaps in their current understandings. Then their packages of mathematical knowledge can be further developed and become more accessible as a mathematical resource for teaching.

**References**


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