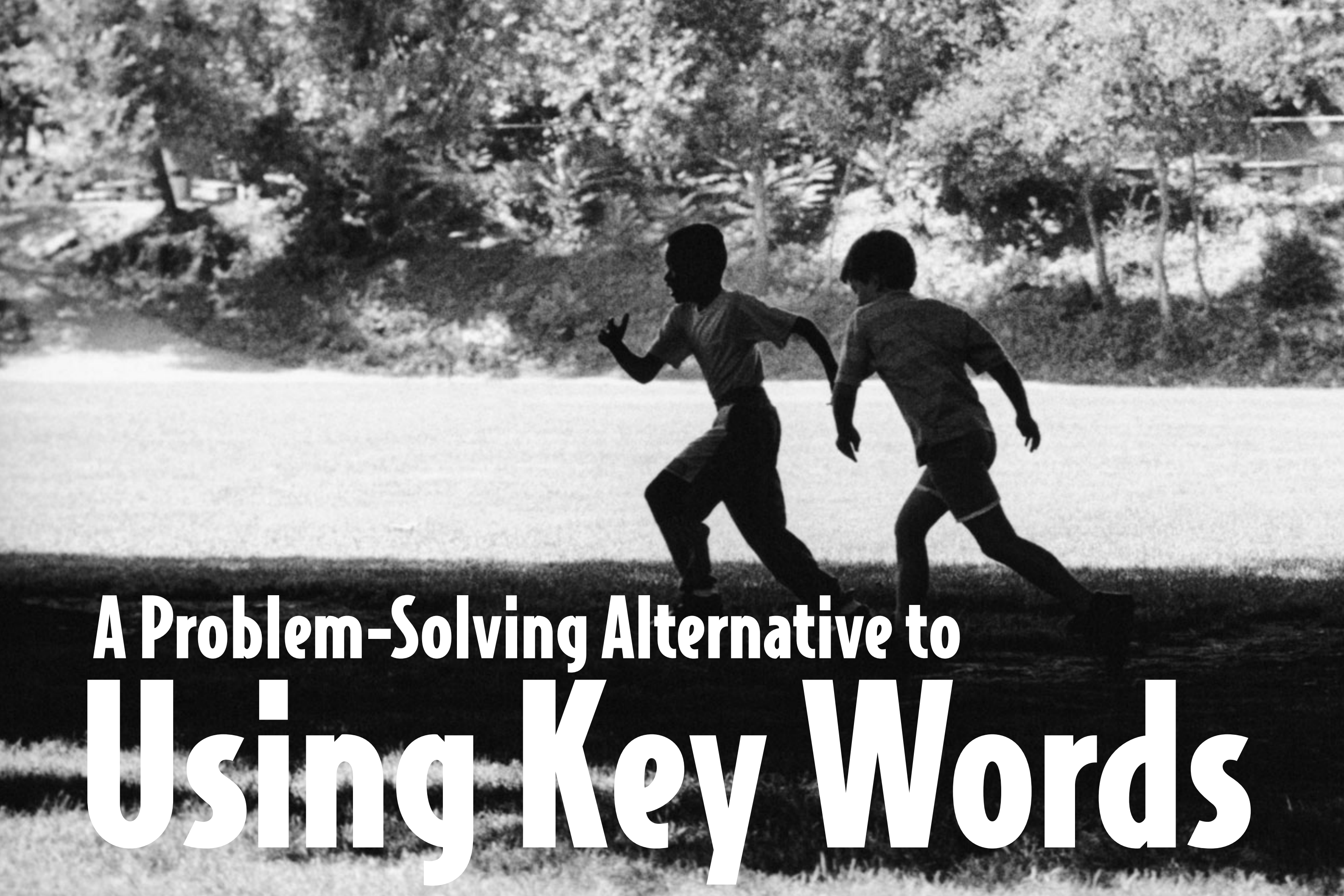

A Problem-Solving Alternative to Using Key Words

LISA L. CLEMENT AND JAMAL Z. BERNHARD

CONSIDER THE FOLLOWING FOUR PROBLEMS.

1. Iliana made 6 times as many jumps when skipping rope as her younger sister, Maria. Maria made 4 jumps. How many jumps did Iliana make?
2. Susan collected 6 rocks, which were 4 more than Jan collected. How many rocks did Jan collect?
3. Elliott ran 6 times as far as Andrew. Elliott ran 4 miles. How far did Andrew run?
4. How many legs do 6 elephants have?

These four problems can be approached in a variety of ways—using a number line, drawing a diagram, using Unifix cubes, and reasoning in one's head about the situation. Often, however, students adopt a key-word approach to solving these types of problems that bypasses any mathematical reasoning (Sowder 1988). Although teachers may introduce key words as a way to *support* students' understanding of situations, students often adopt the key-word approach as a *replacement* for making sense of the situations. For example, in problem 1 a student could successfully use the key-word approach by identifying the word *times* as a signal to multiply 6 and 4. In problems 2–4, however, the key-word strategy fails the student. In problem 2, the key words *more than* incorrectly suggest adding the two numbers together; in problem 3, the key word *times* incorrectly suggests multiplication. In problem 4, only one number is stated explicitly, and there is *no* word in the problem that suggests an operation. In all four situations, the student who uses the key-word approach has adopted a strategy that at times can produce correct answers; however, in this situation, it subverts an approach to problem solving that is supposed to help make sense of the mathematics. This article explores problematic aspects of key words and provides an alternative approach to problem solving that supports students' sense-making.

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Drawbacks to Key Words

WHEN PROBLEMS ARE COMPLEX—THAT IS, WHEN they involve several relationships between and among quantities—the pitfalls with the key-word approach become readily apparent. Consider, for example, the following problem (San Diego State University 1999).

Dieters Problem

Two people who have been on diets are talking:

Dieter A: "I lost 1/8 of my weight—I lost 19 pounds."
Dieter B: "I lost 1/6 of my weight, and now you weigh 2 pounds less than I do."

What was dieter B's original weight?

(Before continuing, you may want to work through the problem and consider how your own students might think about it.)

Although many students are able to at least determine dieter A's weight before the diet, some incorrectly choose to *multiply* 1/8 and 19 because they treat the key word *of* as a signal to multiply. Reasoning like the following is common:

1. 1/8 *of* my weight means 1/8 *times* my weight.
2. 19 pounds is a weight (and the only weight stated in the problem).
3. Thus, I must multiply 1/8 and 19 to find the answer.

Some students also incorrectly choose to *subtract* 2 from dieter A's weight (and sometimes from 1/6) because the key words *less than* obligate them to subtract. After these computations, students are left with a number, but they are unsure what the number represents within the context of the problem. Rather than try to make sense of the situation, students view the problem-solving process as taking a collection of numbers and finding operations to perform, which are based on the key words in the problem, not on understanding the context.

Many students have learned to survive in mathematics classes by memorizing rules and using key words to get answers. However, as soon as problems involve *unfriendly* numbers (for example, problems for which the solution is not a whole number) or more than one operation, these strategies fail them. Students are left with no rule to use and no understanding of the situation, because sense-making was never a part of the students' problem-solving process.

Teachers often present the key-word approach to problem solving with good intentions. For example, we formerly taught key words as a way to help students successfully solve problems. We soon realized, however, that our students were overreliant on key words and used them in lieu of reasoning about or understanding the situation. We thus sought an alternative approach because we wanted our students to develop the conceptual tools to begin making sense of mathematical situations.

Quantitative Analysis

ONE PROMISING ALTERNATIVE TO KEY WORDS IS *quantitative analysis*. Thompson (1994) defined this term as being the process of coming to understand the quantities and relationships between quantities in a mathematical situation. An important aspect of quantitative analysis is the focus on quantities rather than values. A *quantity* is anything that can be measured or counted. The *value* of a quantity is its measure or the number of items that are counted and involves a number and a unit (Thompson 1994). One detrimental feature of the key-word approach is that it tends to focus students' attention primarily on the numbers (or *values)* in the situation and not on the quantities and the relationships between quantities. Although several values occur in the dieter's problem, each value is related to a quantity. Below is a list of some of the quantities that are involved in this situation.

1. Dieter A's original weight
2. Dieter B's original weight
3. Dieter A's new weight after the diet
4. Dieter B's new weight after the diet
5. The amount of weight lost by dieter A (or the difference between dieter A's weight before and after the diet)
6. The amount of weight lost by dieter B (or the difference between dieter B's weight before and after the diet)
7. The difference between dieter A's weight and dieter B's weight before the diets
8. The difference between dieter A's weight and dieter B's weight after the diets
9. The fraction of dieter A's original weight that dieter A lost
10. The fraction of dieter B's original weight that dieter B lost

You may notice that the list of quantities does not, of course, involve any values. You may also be wondering, what is *not* a quantity? Situations that cannot be measured or counted, such as perseverance or joy, are not quantities, although they may be related to this situation. For example, the dieters may have had to persevere to lose the weight, and they may feel much joy at having been successful, but

TABLE 1
Listing Both Quantity and Values That Are Necessary to Solve the Problem

	QUANTITY	VALUE
1	Dieter A's original weight	Could be used to solve the problem
2	Dieter B's original weight	Final value we are seeking
3	Dieter A's new weight after the diet	Needed to solve the problem
4	Dieter B's new weight after the diet	Needed to solve the problem
5	The amount of weight lost by dieter A	Known value, 19 pounds
6	The amount of weight lost by dieter B	Could be used to solve the problem
7	The difference between dieter A's weight and dieter B's weight before the diets	Not needed to solve the problem
8	The difference between dieter A's weight and dieter B's weight after the diets	Known value, 2 pounds
9	The fraction of dieter A's original weight that dieter A lost	Known value, 1/8
10	The fraction of dieter B's original weight that dieter B lost	Known value, 1/6

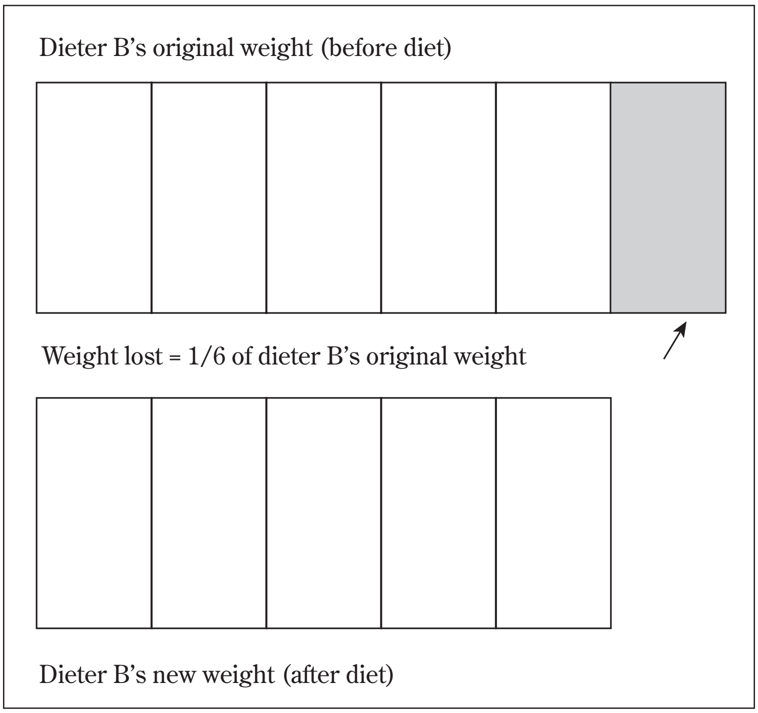

Fig. 1 Drawing a diagram helps students visualize and understand the process behind the mathematics.

there is no way to appropriately assign a measure or count to either *quality,* so they are not *quantities.*

When we return to the list, we recognize that we already know the *values* of some of the quantities because they are given in the problem. **Table 1** includes the list of ten quantities, the known values, and whether knowing the value would be useful in solving the problem. For example, we know that dieter A lost 19 pounds (quantity 5), whereas 1/8 of dieter A's original weight is the fraction of weight that was lost (quantity 9). We need to find the *values* of some of these quantities if we are going to solve the problem (quantities 2, 3, and 4). We do not care about the values of some quantities because they are not relevant to the current problem (quantity 7). All the quantities in the list at left are part of what we call the *quantitative structure* of the situation, even though some may not be relevant to the given problem. The quantitative structure of a problem includes all the quantities and the relationship between and among those quantities.

You may have noticed that quantities 5–10 in our list are made by *combining* or *comparing* two other quantities. For example, the amount of weight lost by the dieters can be found by additively comparing their original with their new weights (using the operation of subtraction), whereas the fraction of weight lost by dieter A can be found by multiplicatively comparing dieter A's weight loss with dieter A's original weight (through the use of a ratio). This example illustrates what is meant by understanding the *relationships between quantities* in a situation. We want our students to realize which quantities are related to one another, and how they are related.

Usefulness of Diagrams

WE HAVE FOUND THAT WHEN MIDDLE-GRADES and older students are prompted to use diagrams to support their reasoning about the quantities in a problem, they are often successful. However, students may be hesitant to use diagrams as a problem-solving approach. Perhaps they feel that drawing pictures is only for young children, or that it is not "mathematical" enough even though using diagrams can be a powerful way to help make sense of the quantitative structure of a situation. If, for example, we wanted a diagram that represented the relationship between dieter B's weight before the diet, dieter B's weight after the diet, the weight lost by dieter B, and the fraction of the weight lost by dieter B, we could draw a diagram like that shown in **figure 1.**

Figure 1 illustrates the fractional relationship between the weight lost by dieter B and the original weight of dieter B, which discerns the relationships between dieter B's weight before and after the diet.

Drawing these types of diagrams not only helps students better understand the quantitative structure of a situation for themselves, but it can also be a powerful communication tool because it allows people to make their thinking explicit to others. For all these reasons, we place a strong emphasis on using diagrams to help analyze mathematical situations quantitatively.

Quantitative Analysis and Selecting Operations

ONCE A STUDENT UNDERSTANDS A SITUATION quantitatively, what to do to solve the problem (that is, the operations to perform) often flows naturally from that understanding. For example, once the relationships embedded in the diagram are understood, one could reason a couple of different ways to find dieter B's initial weight, as shown in the examples that follow:

1. "The original weight is 6/5 as large as the new weight, so we could multiply the new weight by 6/5 to get the original weight."
2. "The weight lost is 1/5 as much as the new weight, so we could divide the new weight by 5 [to determine the weight lost] and add the result to the new weight to get the original weight."

Appropriate operations arise as a result of making sense of the situation's quantitative structure rather than through the use of key words or some other strategy that neglects mathematical understanding. In this problem, key words not only do not support students' understanding, their use actually thwarts it. When using key words, students may react to problems so automatically that they may not consider whether their answers make sense. For example, in problem 3 given at the beginning of the article (Elliott ran 6 times as far as Andrew. Elliott ran 4 miles. How far did Andrew run?), students may quickly get an incorrect answer of 24 (6 times 4) without ever considering the question of "Who ran farther?" If students are oriented toward first identifying quantities, then making sense of the situation (rather than identifying a key word and performing an operation), they may be less likely to make these kinds of mistakes.

Focusing on quantitative analyses gives students an explicit process through which they can begin to make sense of mathematical situations. As students become more familiar with the sense-making process and begin to have more success when thinking about mathematical situations, they will realize what it really means to be a good problem solver. In addition, discussing quantities in the problem (rather than the values) gives students a way to communicate their understanding to others.

Students' Thinking Using the Quantitative Structure

THE LANGUAGE OF QUANTITATIVE REASONING supports *all* students, even those who have previously been successful at problem solving. Successful problem solvers analyze the quantitative structure of a problem as they solve it, which they may do subconsciously. As a result, they often have difficulty reflecting on and explaining their own understanding in a way that can help others or in a way that can help themselves extend their reasoning, because they do not recognize how their sense-making drives their own problem solving. Some successful problem solvers will communicate about *values* rather than *quantities* when explaining their thinking, even though they implicitly focused on the quantities as they solved the problem. The following scenario explores this situation:

Teacher: Who would like to share how they thought about the dieter's problem? Maria?

Maria: First I multiplied 8 times 19 and got 152. Then I subtracted 19 from 152 and got 133. Then I added 133 and 2 to get 135, and then, um, um, [pauses] divided by 5 to get 27 and added 135 and 27 to get 162.

Although Maria's teacher asked her to share how she thought about the problem, Maria was oriented toward sharing the calculations she performed to get the answer (Thompson, Philipp, Thompson, and Boyd 1994). Consider how a teacher might continue to direct this student and her classmates to help them make explicit their reasoning rather than their calculations (by focusing on the quantitative structure of the problem):

Teacher: OK. My guess is that you have done some productive thinking in this problem but that the students in the class who did not do the exact same calculations might not know *why* you did *what* you did. Let's back up and talk about the problem again. Before you tell me what operations you performed, can anyone in the class tell me what this problem was about?

Ellen: There are two dieters who lost weight.

Teacher: Can you tell me anything else about the problem?

Maria: After the diet, dieter A weighed less than dieter B.

Teacher: And how do you know that?

Jose: Because dieter B said that dieter A now weighs 2 pounds less than dieter B.

Teacher: What other relationships do you notice?

Anna: Dieter B lost a greater fraction of her original weight than dieter A.

Teacher: And how do you know?

Anna: Because dieter B lost 1/6 of her weight, but dieter A lost only 1/8 of her weight.

Teacher: OK, so we have this problem about two dieters who have lost some weight, and we know something about who lost more weight, and we know something about who lost a greater fraction of their original weight.

At this point, the teacher and the students are not focused on getting a numerical answer (although that is one of their eventual goals). They are talking about the dieters and some relationships between their weights, so that they can better understand what the problem is about. Talking about the quantities changes the focus from finding a solution to understanding relationships. The teacher could then continue in this way:

Teacher: Maria, let's go back and think about your approach to solving the problem. You began by multiplying 8 and 19. Why 8 and 19? What did they represent?

Maria: I knew that dieter A lost 19 pounds, and that 19 pounds was 1/8 of her original weight. I knew that I could get her original weight [before the diet] by multiplying the weight she lost by 8.

Teacher: Hmmm. . . . [Pretending to be perplexed.] I see 1/8 in the problem, not 8, so why did you multiply the weight dieter A lost by 8?

Judith: I have a picture that might help. I drew a picture to show dieter A's original weight and the weight she lost. [Judith draws the diagram shown in **fig. 2.**] I cut the rectangle that shows dieter A's original weight into 8 equal pieces, because she lost 1/8 of her weight. Because 1/8 of her weight was 19 pounds, I knew that every 1/8 of her weight was equal to 19 pounds, so I multiplied 19 times 8 to get her original weight.

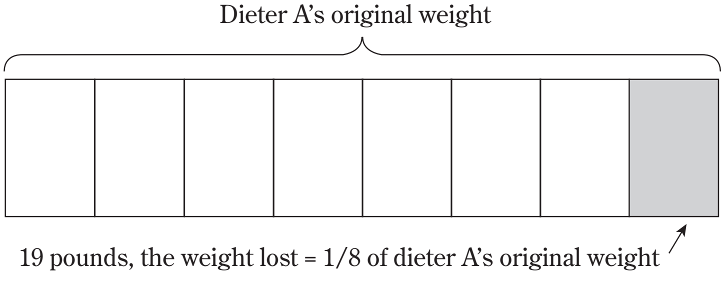

Fig. 2 This diagram explores dieter A's original and new weight, to aid understanding.

Students like Maria can answer a problem correctly but often have difficulty articulating *why.* When students take the time to explicitly analyze the quantitative structure of the situation, they can see exactly what was understood about the problem that led to their solution. In so doing, they can also communicate this understanding to others. In contrast, a key-word approach focuses on operating on numbers without understanding the underlying reasons for the calculations.

Instructional Implications

STUDENTS CAN BE ENCOURAGED TO ASK THEMselves the following questions when analyzing a mathematical situation quantitatively:

- What quantities are involved in this situation?
- What quantities am I trying to find?
- Which quantities are critical to the problem at hand?
- Are any of these quantities related to each other? If so, how are they related?
- Do I know the values of any of the quantities? Which ones?
- For which quantities do I *not* know the value? Are these quantities related to other quantities in the situation? Can these relationships help me find any unknown values?
- Would drawing a diagram or enacting the situation help me to answer any of these other questions (San Diego State University 1999)?

These questions are meant to help students make explicit the quantitative structure of the situation and are meant to help them initially focus on the relationships between quantities in the problem rather than on the values.

Summary

IN THIS ARTICLE, WE IDENTIFIED PROBLEMS with using a key-word approach to support students' problem-solving efforts. The use of key words subverts mathematical understanding, can lead to incorrect solutions, focuses students' attention on values rather than quantities, and orients students toward automatically performing procedures rather than first making sense of the situation. Because this method has limited utility, it can also prevent students from making appropriate generalizations.

We have introduced an alternative approach, the idea of quantitative analysis (Thompson 1994), as the process of coming to understand the quantities and relationships between quantities in a situation, and demonstrated what we mean by using the

Dieters problem. This practice of explicitly trying to understand the quantitative structure of a situation provides students with a process for making sense of mathematics, helps them to reflect on their own thinking and make it explicit, gives them the tools to communicate with others about their understanding, and forces them to think about what it means to be a good problem solver. Ample evidence shows that even young children can and do reason quantitatively when given opportunities to do so and that the emphasis on reasoning quantitatively (rather than searching for keys words to help determine the appropriate operation) helps students to advance their mathematical thinking about situations (Hiebert, Carpenter, Fennema, Fuson, Wearne, and Murray 1997).

Initially, students may need to make explicit each piece of performing a quantitative analysis (for example, explicitly writing down all of the quantities in a problem), although eventually they will internalize and abstract these pieces. The ultimate goal is *not* for students to list all the quantities in a situation before solving every problem. However, because many students do not know how to begin making sense of mathematical situations, we initially support our students' understanding by asking them to be explicit about the aspects of a situation that are important. The idea of quantitative analysis serves this role well. We hope that *all* our students will develop the ability to think about the quantitative structure of mathematical situations. If this ability develops, however, it must stem from the understanding that the focus should always be on *making sense* of mathematical situations (which involves thinking about quantities), and not just on what key words suggest that they *do*.

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